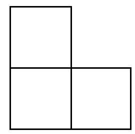
CMC 2020 Solutions

Problem 1

Consider an $n \times n$ unit-square board. The main diagonal of the board is the n unit squares along the diagonal from the top left to the bottom right. We have an unlimited supply of tiles of this form:

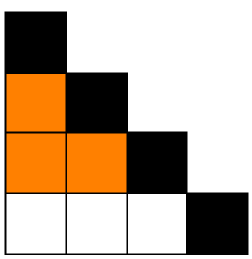


The tiles may be rotated. We wish to place tiles on the board such that each tile covers exactly three unit squares, the tiles do not overlap, no unit square on the main diagonal is covered, and all other unit squares are covered exactly once. For which $n \geq 2$ is this possible?

Solution

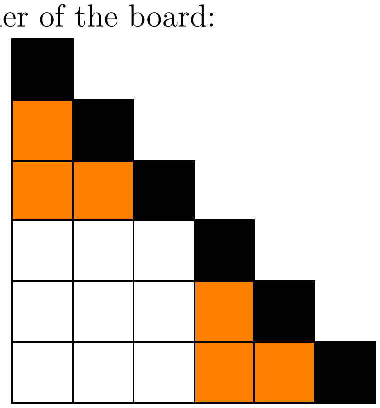
The board consists of N^2 unit squares, of which N should not be covered. Each tile covers exactly three squares, so we must have $3 \mid N(N-1)$. Hence if $N \equiv 2 \mod 3$, the board cannot be covered. From now on we will only consider $N \equiv 0, 1 \mod 3$.

The board can easily be covered for $N = 3$. For $N = 4$, the picture below shows the main diagonal in black and the bottom left corner of the board:



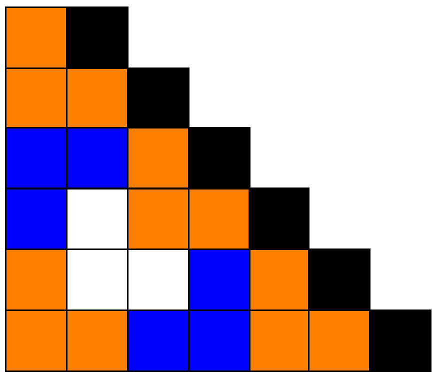
In order to cover the unit square in the top left corner, one of the tiles must be placed on the orange squares. However, then the other unit squares in this half of the board cannot be covered any more. So $N = 4$ is not possible.

For $N = 6$, the picture below shows the main diagonal in black and the bottom left corner of the board:

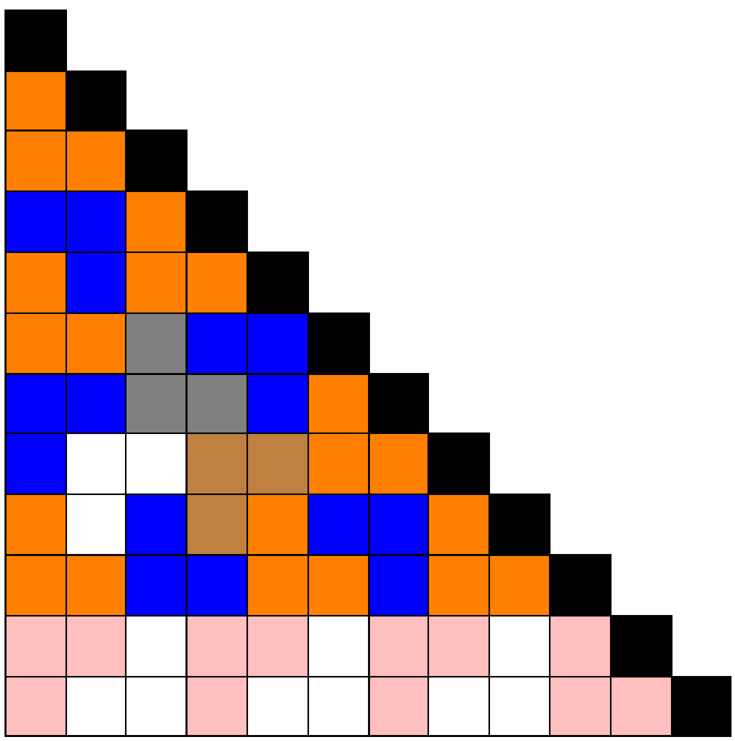


Again, two tiles must be places exactly on the orange unit squares. For the 3×3-board that is left, we need three tiles; however, each tiles covers at most one of the four corners. So this is impossible.

Now we will show that all other $N \equiv 0, 1 \mod 3$ are possible. The picture below shows a solution for $N = 7$, where the second half of the board can be filled similarly:



And the next picture shows a solution for $N = 10$, including a way of extending this solution to $N = 12$.



In general, for $N = 3k + 1$ we can extend the solution to $N = 3k + 3$ by adding two rows on the bottom and putting a tile on the far right of those two rows. Then we have a rectangle of size $2 \times 3k$ left, which we can cover by creating a 2×3-rectangle of two tiles, and putting k of those next to each other. So if $N = 3k + 1$ is possible, then $N = 3k + 3$ is possible.

Also, we can extend the solution for $N = 3k + 1$ to a solution for $N = 3k + 7$ by adding six rows to the bottom. Then on the far right of these rows, we can put the construction for

$N = 7$ (see picture above). Then we have a rectangle of size $6 \times 3k$ left, which we can cover by using $3 \times k$ of the 2×3-rectangles consisting of two tiles.

So starting from $N = 7$ and $N = 10$ we can find constructions for all $N \equiv 1 \mod 3$ with $N \geq 7$, and from those we can find constructions for all $N \equiv 0 \mod 3$ for $N \geq 9$. We conclude that the N which are possible are $N = 3$, $N \equiv 0 \mod 3$ with $N \geq 9$, and $N \equiv 1 \mod 3$ with $N \geq 7$.

Problem 2

Let $f(x) = 3x^2 + 1$. Prove that for any given positive integer n, the product

$$f(1) \cdot f(2) \cdot \dots \cdot f(n)$$

has at most n distinct prime divisors.

Solution

Call a prime divisor p of $f(n)$ *new* if p does not divide any of $f(1), \dots, f(n-1)$.

Consider a new prime divisor p of $f(n)$. Clearly, $p \neq n$, because then p does not divide $3n^2 + 1$. Note that if $p < n$, then $1 \leq n - p < n$ and $f(n-p) \equiv f(n) \equiv 0 \pmod{p}$, contradicting to the assumption that p is new. If $n < p < 2n$, then $1 \leq p - n < n$ and $f(p-n) \equiv f(n) \equiv 0 \pmod{p}$, contradicting to the same assumption. It follows that $p \geq 2n \geq \sqrt{f(n)}$.

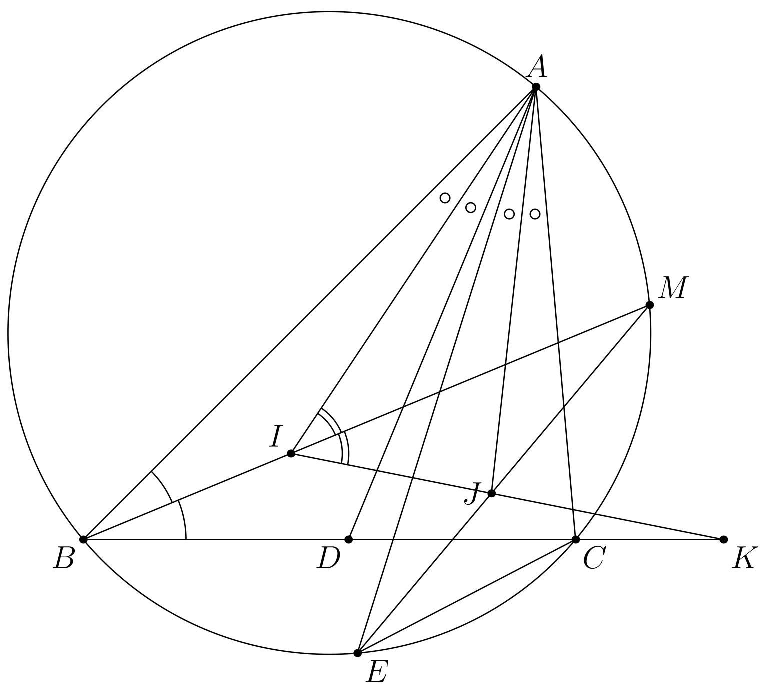
The number $f(n)$ cannot have two distinct prime divisors greater than or equal to its square root. Therefore, $f(n)$ has at most one new prime divisor, and the problem statement follows by induction.

Problem 3

Let ABC be a triangle such that $AB > BC$ and let D be a variable point on the line segment BC. Let E be the point on the circumcircle of triangle ABC, lying on the opposite side of BC from A such that $\angle BAE = \angle DAC$. Let I be the incenter of triangle ABD and let J be the incenter of triangle ACE. Prove that the line IJ passes through a fixed point, that is independent of D.

Solution 1

(by Géza Kós) If point D approaches point B, then so does point I and point J approaches point C, so the desired common point must lie on line BC. Let $K = BC \cap IJ$. We prove that K is a fixed point.



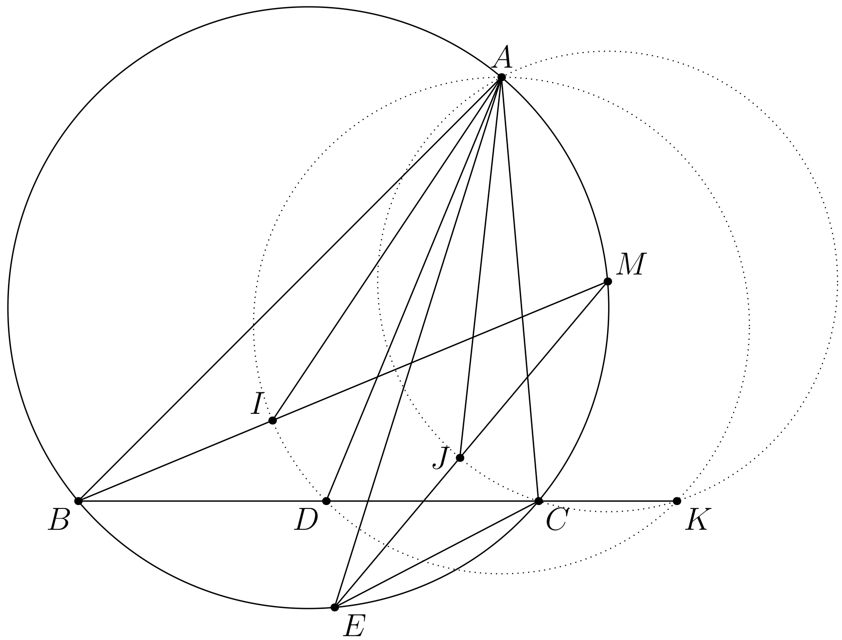
Let M be the midpoint of the arc $\widehat{AC}$ of the circumcircle. The angle bisectors of $\angle ABC$ and $\angle AEC$, BI and EJ, pass through point M. It is well-known that $MA = MC = MJ$. Rays AI and AJ bisect congruent angles $\angle BAD$ and $\angle EAC$. It follows that $\angle IAJ = \angle BAE = \angle BME = \angle IMJ$, so $AIJM$ is a cyclic quadrilateral. Because $MA = MJ$, line IM bisects $\angle AIJ$.

Finally, note that triangles AIB and KIB are congruent. Thus K is the reflection of A about the fixed line BM, so K is a fixed point as desired.

Solution 2

(by Merlijn Staps) We show that the fixed point is the point K on the extension of BC beyond C satisfying $BK = BA$.

Note that because K is the reflection of A over BI, triangles AIB and KIB are congruent. It follows that $\angle IKB = \angle BAI = \angle IAD$, yielding $AIDK$ is a cyclic quadrilateral.



On the other hand, because J is the incenter of triangle AEC, we have

$$\angle CJA = 90^\circ + \frac{1}{2}\angle CEA = 90^\circ + \frac{1}{2}\angle KBA = 180^\circ - \angle BKA = 180^\circ - \angle CKA.$$

Hence $AJCK$ is a cyclic quadrilateral. We now have

$$\angle JKC = \angle JAC = \tfrac{1}{2}\angle EAC = \tfrac{1}{2}\angle BAD = \angle IAD = \angle IKD,$$

from which it follows that IJ passes through K.

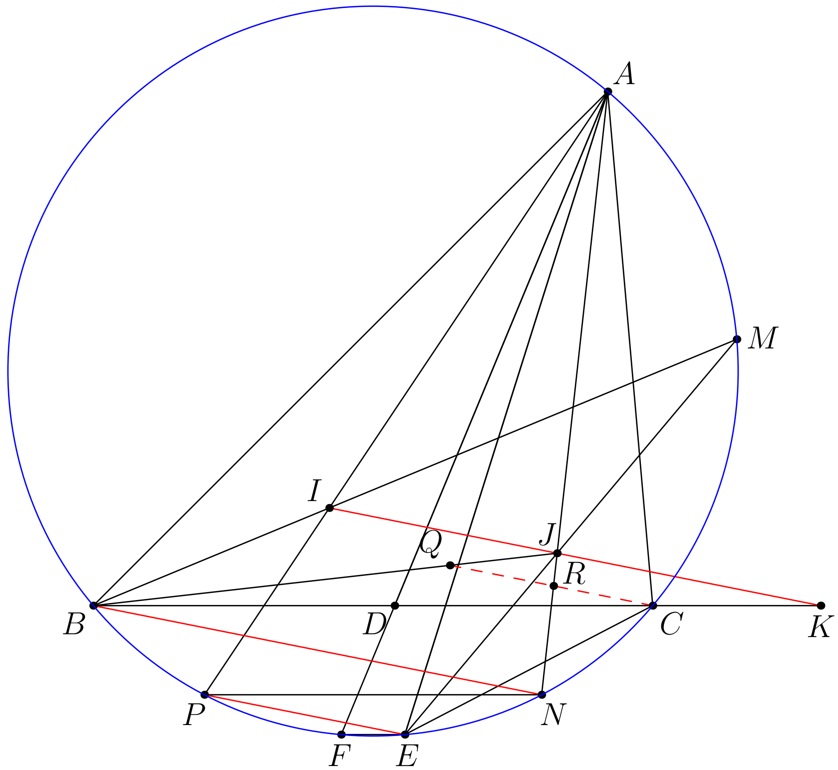
Solution 3

(by Alex Zhai) Since $\angle ABD = \angle AEC$ and $\angle BAD = \angle EAC$, we have a spiral similarity centered at A taking ACE to ADB, which also takes J to I as they are the incenters of the respective triangles. Therefore, triangles ACD, AJI and AEB are similar and $\angle AJI = \angle ACB$.

Let K be the intersection of lines IJ and BC. Because $\angle AJK = 180^\circ - \angle AJI = 180^\circ - \angle ACB = \angle ACK$, we conclude that quadrilateral $AJCK$ is cyclic. Denote by M be the midpoint of arc AC in the circumcircle of ABC. It is well-known that $MA = MC = MJ$. Thus the circumcircle of quadrilateral $AJCK$ is a fixed circle centered at M and therefore it intersects line BC in fixed points C and K.

Solution 4

(by Evan Chen) Let lines BC and IJ meet at K. Extend rays AI, AD, AJ, BI to meet the circumcircle again at P, F, N, M.



We use Pascal's theorem to points A, B, E and M, P, N lying on the circumcircle of triangle ABC. Points $I = AP \cap BM$, $J = AN \cap EM$, and intersection of the lines $BN \cap EP$ must be collinear. Note that $\angle BAP = \angle EAN$, so $BPEN$ is an isosceles trapezoid and $BN \parallel PE$. It follows that the intersection of $BN \cap PE$ is a "point at infinity" and as a consequence, from Pascal's Theorem, we must have $IJ \parallel BN \parallel PE$.

Define points Q and R on lines BJ and NJ such that QR passes through C and is parallel to the three lines we have found. Using the classical fact that $JN = CN$, we have

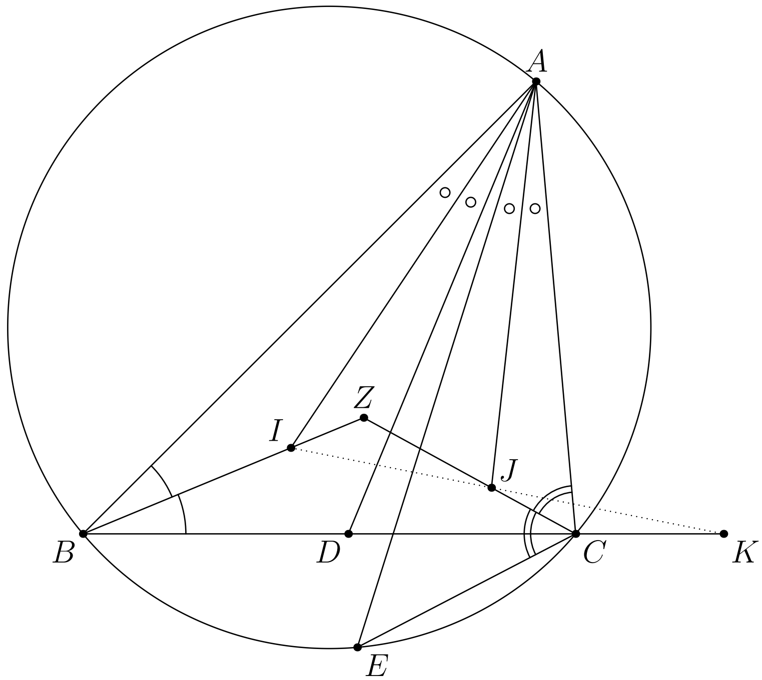
$$\frac{BK}{BC} = \frac{BJ}{BQ} = \frac{JN}{RN} = \frac{CN}{RN} = \frac{\sin \angle NRC}{\sin \angle RCN} = \frac{\sin \angle ANB}{\sin \angle BNC} = \frac{AB}{BC}.$$

Therefore $BK = AB$ and K is a fixed point, as desired.

Solution 5

(by Ivan Borsenco) Denote $\angle BAD = \angle EAC = 2\theta$. Let BI and CJ intersect in point Z. We have $\angle ECJ = \angle ACJ = \frac{A}{2} + \frac{C}{2} - \theta = 90 - \frac{B}{2} - \theta$ and

$$\angle BCJ = \angle ECJ - \angle ECB = \frac{A}{2} + \frac{C}{2} - \theta - (A - 2\theta) = \frac{C}{2} - \frac{A}{2} + \theta.$$



In order to show that point K that lies on BC is fixed, we prove that $\dfrac{CK}{BK}$ is fixed.

From Menelaus Theorem we have $\dfrac{BI}{ZI} \cdot \dfrac{ZJ}{CJ} \cdot \dfrac{CK}{BK} = 1$. Note that from the Law of Sines in triangles AIB and CJA, we get

$$\frac{BI}{\sin\theta} = \frac{AB}{\sin(\frac{B}{2}+\theta)}, \quad \frac{CJ}{\sin\theta} = \frac{AC}{\sin(90^\circ + \frac{B}{2})}$$

and

$$BI = \frac{2R\sin C\sin\theta}{\sin(\frac{B}{2}+\theta)}, \quad CJ = \frac{2R\sin B\sin\theta}{\cos(\frac{B}{2})}, \quad \frac{BI}{CJ} = \frac{\sin C}{2\sin\frac{B}{2}\sin(\frac{B}{2}+\theta)}.$$

Also, from the Law of Sines in triangle BZC, we get

$$\frac{BZ}{\sin(\frac{C}{2}-\frac{A}{2}+\theta)} = \frac{CZ}{\sin\frac{B}{2}} = \frac{BC}{\sin(90^\circ + A - \theta)}.$$

It follows that

$$ZI = BZ - BI = \frac{2R\sin A\sin(\frac{C}{2}-\frac{A}{2}+\theta)}{\cos(A-\theta)} - \frac{2R\sin C\sin\theta}{\sin(\frac{B}{2}+\theta)}.$$

$$ZJ = CZ - CJ = \frac{2R\sin A\sin\frac{B}{2}}{\cos(A-\theta)} - \frac{2R\sin B\sin\theta}{\cos(\frac{B}{2})}.$$

We have

$$\begin{aligned}2\sin(\frac{C}{2}-\frac{A}{2}+\theta)\sin(\frac{B}{2}+\theta) &= \cos\left(\frac{C}{2}-\frac{A}{2}+\theta-(\frac{B}{2}+\theta)\right) - \cos\left(\frac{C}{2}-\frac{A}{2}+\theta+(\frac{B}{2}+\theta)\right)\\ &= \cos(C-90^\circ) - \cos(90^\circ - A + 2\theta)\\ &= \sin(C) - \sin(A-2\theta),\end{aligned}$$

and $2\sin\theta\cos(A-\theta) = \sin A + \sin(2\theta - A) = \sin\theta - \sin(A-2\theta)$. Therefore

$$\begin{aligned} ZI &= R\cdot\frac{\sin A\,(\sin(C)-\sin(A-2\theta)) - \sin C(\sin A - \sin(A-2\theta))}{\cos(A-\theta)\sin(\frac{B}{2}+\theta)} \\ &= \frac{R\sin(A-2\theta)(\sin C - \sin A)}{\cos(A-\theta)\sin(\frac{B}{2}+\theta)}, \\ ZJ &= \frac{2R\sin\frac{B}{2}\,(\sin A - 2\sin\theta\cos(A-\theta))}{\cos(A-\theta)} = \frac{2R\sin\frac{B}{2}\sin(A-2\theta)}{\cos(A-\theta)}, \end{aligned}$$

and

$$\frac{ZJ}{ZI} = \frac{2\sin\frac{B}{2}\sin(\frac{B}{2}+\theta)}{\sin C - \sin A}.$$

Using the above results, we get

$$\frac{BI}{ZI}\cdot\frac{ZJ}{CJ} = \frac{BI}{CJ}\cdot\frac{ZJ}{ZI} = \frac{\sin C}{2\sin\frac{B}{2}\sin(\frac{B}{2}+\theta)}\cdot\frac{2\sin\frac{B}{2}\sin(\frac{B}{2}+\theta)}{\sin C - \sin A} = \frac{\sin C}{\sin C - \sin A} = \frac{c}{c-a},$$

yielding $\dfrac{CK}{BK} = \dfrac{c-a}{c}$, so point K is fixed.

Solution 6

(by Mehtaab Sawhney) We proceed via a direct use of complex numbers. Let the circumcircle of ABC be the unit circle and assign coordinate $A = a^2$, $B = b^2$, and $C = c^2$. Furthermore let L denote the intersection of AD with the circumcircle of (ABC) and set $L = \ell^2$. It is well-known that signs for a, b, c, ℓ can be chosen so that $M_{AB} = -ab$, $M_{AC} = -ac$, $M_{BL} = b\ell$, and $M_{LC} = -\ell c$ where M_{AB} denotes the midpoint of the arc AB not containing C and similar for the remaining points. Finally let O denote the circumcenter and note that it has coordinate 0.

Now since $\angle COE = \angle LOB$ we fine that $E = \frac{b^2c^2}{\ell^2}$. Through similar angle chasing we find that $M_{CE} = \frac{bc^2}{\ell}$ and $M_{AE} = \frac{abc}{\ell}$. Since the in-center of a triangle is the vector sum of arc midpoints if the circumcenter is at the origin we find that

$$J = \frac{abc}{\ell} - ac + \frac{bc^2}{\ell}.$$

Now using that $I = BM_{AC} \cap AM_{BL}$ and the chord intersection formula we find that

$$I = \frac{a\ell(b^2-ac) + bc(a^2+b\ell)}{a\ell + bc}.$$

The first identity is to realize that $I - J$ has a particularly nice form, in particular

$$I - J = \frac{(\ell^2 - c^2)(a+c)b^2}{\ell(bc + a\ell)}.$$

Now suppose there exists S denote the point which lies on IJ independent of ℓ; this point would satisfy

$$\frac{S-J}{\overline{S}-\overline{J}} = \frac{I-J}{\overline{I}-\overline{J}} = \frac{-b^3c^2}{\ell}.$$

We now "cheat" and get an additional equation from the case when $\ell = b$; in this degenerate case $I = b$, $J = c$ and thus such a point S would lie on BC and hence satisfies

$$S + b^2c^2\overline{S} = b^2 + c^2.$$

Thus we have a pair of equations in $S, \overline{S}$ and solving gives

$$S = b - ac + \frac{b^2c}{a}.$$

This finishes noting that we have proved for arbitrary L that I, J intersects the point S which is clearly independent of the choice of ℓ.

Problem 4

Let n be an odd positive integer. Some of the unit squares of an $n \times n$ unit-square board are colored green. It turns out that a chess king can travel from any green unit square to any other green unit squares by a finite series of moves that visit only green unit squares along the way. Prove that it can always do so in at most $\frac{n^2-1}{2}$ moves. (In one move, a chess king can travel from one unit square to another if and only if the two unit squares share either a corner or a side.)

Solution

We refer to the n^2 unit squares in the grid by their coordinates (x, y) for $0 \le x \le n-1$ and $0 \le y \le n-1$. Define the *king graph* G_n to be the graph with vertices corresponding to the unit squares, where two vertices (x_1, y_1) and (x_2, y_2) are joined by an edge if and only if the chess king can move between them (i.e. $\max(|x_1 - x_2|, |y_1 - y_2|) = 1$).

We define a *snake-in-the-box* path of length ℓ in G_n to be a sequence $a_1, a_2, \ldots, a_{\ell+1}$ such that a_i and a_j are joined by an edge of G_n if and only if $|i - j| = 1$. For any two green unit squares p and q, consider the minimal length path from p to q in the graph G_n, where all vertices in the path correspond to green unit squares. Because such a path has minimal length, it is necessarily a snake-in-the-box path. Thus, the problem reduces to (and is in fact equivalent to) showing that the length of any snake-in-the-box path in G_n is at most $(n^2-1)/2$.

We define a vertex (x, y) of G_n to be *special* if both x and y are odd, and *regular* otherwise. Given a snake-in-the-box path $\mathcal{P}$, let $S(\mathcal{P})$ be the number of special vertices in the path, and let $R(\mathcal{P})$ be the number of edges in $\mathcal{P}$ connecting two regular vertices (we say such an edge is a *regular edge*). Since each special vertex is connected to at most two edges in $\mathcal{P}$, we see that the length of $\mathcal{P}$ is at most $2S(\mathcal{P}) + R(\mathcal{P})$.

To bound the quantity $2S(\mathcal{P}) + R(\mathcal{P})$, we cover the special vertices and regular edges in G_n with a number of *blocks* (small sets of vertices and edges). The blocks will be of two types:

- A *small block* consists of 4 vertices, of which exactly 1 is special, and 3 regular edges. The vertices are of the form $\{(x+i, y+j) : 0 \le i, j \le 1\}$. Note that exactly one of these vertices must be special. The edges are those connecting the remaining three regular vertices in the block, as illustrated in Figure 1a.

- A *large block* consists of 9 vertices, of which exactly 2 are special, and 8 regular edges. We describe one possible orientation of a large block: the vertices are of the form $\{(x+i, y+j) : 0 \le i, j \le 2\}$, where x is even and y is odd. Thus, for this orientation $(x+1, y)$ and $(x+1, y+2)$ are the two special vertices. The edges are all the regular edges incident to $(x, y+1)$ or $(x+1, y+1)$, as illustrated in Figure 1b. We also allow three other orientations obtained by 90-degree rotations.

Lemma. Write $n = 2k+1$. There exists a set of $4\left\lceil \frac{k}{2} \right\rceil$ small and $4\left\lfloor \frac{k^2}{4} \right\rfloor$ large blocks such that each special vertex is in at least 2 blocks, and each regular edge is in at least one block.

(a) A small block.

(b) A large block.

Figure 1: For both the small and large blocks, the figures show only one of four possible orientations. The white dots are a visual aid to help identify the blocks in diagrams that follow.

Proof. We give an explicit construction. The construction is most easily understood by looking at Figures 2 and 3 for the cases $n = 13$ and $n = 15$, respectively.

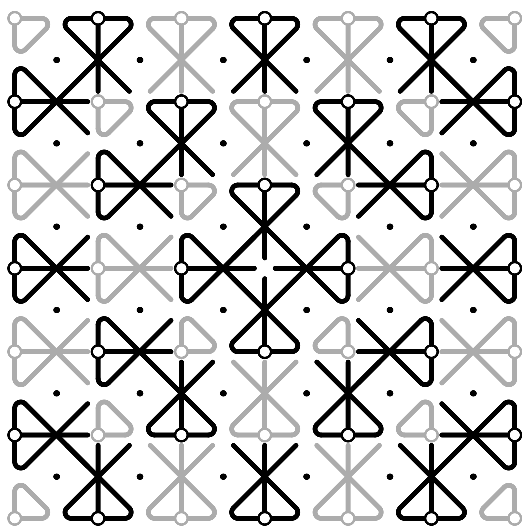


Figure 2: Block covering for $n = 13$ with 12 small blocks and 36 large blocks. Blocks are shown in two colors as a visual aid; the coloring has no bearing on the solution.

To describe it more formally, for each $a < k$ where a is even, we place a small block with vertices $\{(a+i, a+j) : 0 \leq i, j \leq 1\}$, with $(a+1, a+1)$ being the special vertex. There are $\lceil \frac{k}{2} \rceil$ such small blocks.

For each (x, y) where x and y are both even, $x > y$, and $x + y < 2k$, we place a large block with vertices $\{(x-1+i, y+j) : 0 \leq i, j \leq 2\}$, where $(x-1, y+1)$ and $(x+1, y+1)$ are the two special vertices (and the edges are incident to (x, y) and $(x, y+1)$). There are

$$(k-1) + (k-3) + \cdots = \left\lfloor \frac{k^2}{4} \right\rfloor$$

such large blocks.

Finally, we repeat this construction three more times by applying 90-degree rotational symmetry for the grid. This yields the desired total count of $4 \lceil \frac{k}{2} \rceil$ small blocks and $4 \left\lfloor \frac{k^2}{4} \right\rfloor$ large blocks.

Furthermore, it can be easily seen in the construction that each special vertex belongs to at least 2 blocks (when k is odd, the vertex (k, k) is in 4 small blocks), and each regular edge belongs to at least one block (some regular edges crossing the main diagonals of the grid belong to 2 blocks).

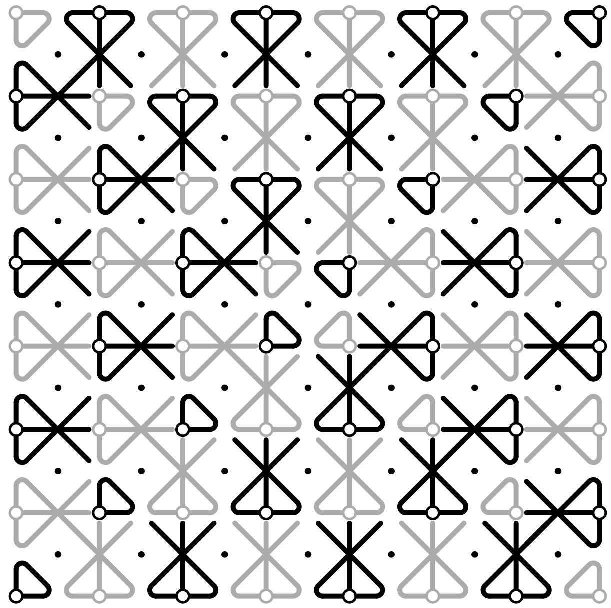


Figure 3: Block covering for $n = 15$ with 16 small blocks and 48 large blocks. Blocks are shown in two colors as a visual aid; the coloring has no bearing on the solution.

We can use the above construction to bound $S(\mathcal{P})$ and $R(\mathcal{P})$, with the help of the following lemma.

Lemma. Let $\mathcal{P}$ be a snake-in-the-box path. In any small block, the total number of special vertices and regular edges in $\mathcal{P}$ is at most 1. For any large block, this number is at most 2.

Proof. The claim can be verified by straightforward caseworks. Let us call a vertex or edge in a block *active* if it is in $\mathcal{P}$ and *inactive* otherwise. Define the *score* of the block to be the number of active special vertices plus the number of active regular edges.

For the small block, note that because $\mathcal{P}$ is a snake-in-a-box path, at most two vertices can be active. Thus, at most one regular edge can be active, and if there is one such edge, both vertices are regular and so no special vertex is active. We conclude that the score is at most 1.

For the large block, suppose we translate and orient it so that the vertices are $\{(1+i, j) : 0 \le i, j \le 2\}$, so that $(1, 1)$ and $(3, 1)$ are the special vertices, and the edges are the regular edges incident to $(2, 0)$ and $(2, 1)$.

If $(2, 1)$ is active, note that $(2, 1)$ is adjacent to all other vertices in the block. Thus, at most two of the other vertices are active, and these two cannot be adjacent to each other. If v is a vertex other than $(2, 1)$ that is active, then it can contribute to the block's score by being a special vertex or sharing a regular edge with $(2, 1)$. But these two cases are mutually exclusive, so the total score of the block is at most 2.

If $(2, 1)$ is inactive, then the only regular edges that can be active are $(1, 0)$—$(2, 0)$ and $(2, 0)$—$(3, 0)$. But at most one of the special vertex $(1, 1)$ and the regular edge $(1, 0)$—$(2, 0)$ can be active, and similarly at most one of $(3, 1)$ and $(2, 0)$—$(3, 0)$ can be active. Thus, the total score is again at most 2.

Since each special vertex is covered at least twice and each regular edge is covered at least once, the quantity $2S(\mathcal{P}) + R(\mathcal{P})$ can be bounded by the sum of scores over all blocks. Hence we have

$$\begin{aligned} \operatorname{length}(\mathcal{P}) &\leq 2S(\mathcal{P}) + R(\mathcal{P}) \leq (\#\text{ small blocks}) + 2\,(\#\text{ large blocks}) \\ &= 4\left\lceil \frac{k}{2} \right\rceil + 8\left\lfloor \frac{k^2}{4} \right\rfloor = 2k + 2k^2 = \frac{n^2-1}{2}, \end{aligned}$$

which is the desired bound.

Remark. The given bound is sharp. One snake-in-the-box path in G_n of length $(n^2-1)/2$ is the zigzag $(0,0) \to (n-2,0)$—$(n-1,1)$—$(n-2,2) \to (1,2)$—$(0,3)$—$(1,4) \to (n-2,4) \cdots$, and so on, where $a \to b$ denotes a subpath which proceeds in a straight line from a to b.

Remark. Snake-in-the-box paths in king graphs for even n are considered in Donald Knuth, The Art of Computer Programming, volume 4, section 7.2.2.1, Dancing Links, exercise 172. However, knowledge of that exercise does not help with the present problem.

Remark. The term "snake-in-the-box" was first introduced by William Kautz in Unit-Distance Error-Checking Codes, IRE Transactions on Electronic Computers, 1958, volume EC-7, in relation to paths in hypercube graphs.

Problem 5

There are 2020 positive integers written on a blackboard. Every minute, Zuming erases two of the numbers and replaces them by their sum, difference, product, or quotient. For example, if Zuming erases the numbers 6 and 3, he may replace them with one of the numbers in the set $\{6+3, 6-3, 3-6, 6\times 3, 6\div 3, 3\div 6\} = \{9, 3, -3, 18, 2, \frac{1}{2}\}$. After 2019 minutes, Zuming arrives at the single number -2020 on the blackboard. Show that it is possible for Zuming to have arrived at the single number 2020 on the blackboard instead, under the same rules and using the same 2020 starting integers.

Solution 1

We show that if Zuming's original set of moves leads to a state $a_1, a_2, \ldots, a_n$ for some $n \le 2020$, he can make new moves to lead to a state $|a_1|, |a_2|, \ldots, |a_n|$. This clearly implies the problem by taking $n = 1$.

We do this by downwards induction. Note that since all starting numbers are positive, the base case $n = 2020$ trivially holds. For the inductive step (from n to $n-1$), it suffices to show that if it is possible to obtain c from a and b, then it is possible to obtain $|c|$ from $|a|$ and $|b|$. If the operation to get c from a and b is multiplication or division, then using the same operation on $|a|$ and $|b|$ will work. Now suppose the operation is addition or subtraction (i.e. $c \in \{a+b, a-b, b-a\}$), then depending on whether a and b are negative or non-negative, we have $c \in \{|a|+|b|, |a|-|b|, -|a|+|b|, -|a|-|b|\}$. And hence $|c| \in \{|a|+|b|, |a|-|b|, |b|-|a|\}$, so we can always choose the appropriate addition or subtraction to get $|c|$ from $|a|$ and $|b|$.

Solution 2

We directly construct a new sequence of moves from Zuming's original sequence as the following:

- Consider the last time that Zuming replaces two numbers with their difference (which must occur since the final number is negative while the starting numbers are all positive), and perform all the moves before then as before. Call this last subtraction move *critical*.

- Instead of replacing a and b with $a-b$ in this critical move, replace them with $b-a$, and call the resulting number (in both the old and new sequence of moves) *special*.

- From now on, every time Zuming replaces the special number s and some other number n with $s+n$, sn, $\frac{s}{n}$, or $\frac{n}{s}$ in the old sequence (note that there are no more subtractions after the critical move), he now replaces the special number s' and the other number n with $s'-n$, $s'n$, $\frac{s'}{n}$, or $\frac{n}{s'}$ respectively. Note that if $s' = -s$, then the resulting numbers in the two sequences are also opposite in sign. Call this new number (in both sequences) special, and repeat. (If Zuming replaces two non-special numbers, then the same move is performed in the new sequence.)

Observe that starting with the critical move, there is exactly one special number on the blackboard at all times, and the two special numbers in the old and new sequences are always exactly opposite in sign since each move involving them preserve this property. Therefore, the last number on the board after 2019 minutes must be special, and hence Zuming will arrive at 2020 in the new sequence.

Problem 6

Find all integers $n \geq 3$ for which the following statement is true: If $\mathcal{P}$ is a convex n-gon such that $n-1$ of its sides have equal length and $n-1$ of its angles have equal measure, then $\mathcal{P}$ is a regular polygon. (A *regular* polygon is a polygon with all sides of equal length, and all angles of equal measure.)

Solution

First part. We first construct a counterexample for all odd $n = 2k-1$. If $k = 2$, we construct an isosceles, non-equilateral triangle; so assume $k \geq 3$.

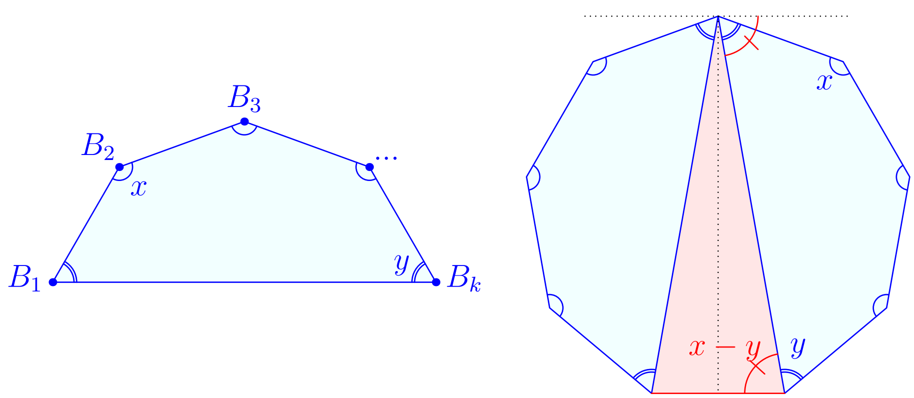


Figure 4: Construction for odd n.

The construction proceeds in the following three steps; see Figure 4.

- Pick a real number $\frac{k-2}{k-1} \cdot 180° < x < \frac{k-1}{k} \cdot 180°$, and choose $y = \frac{(180°-x)(k-2)}{2}$ such that

  $$(k-2)x + 2y = (k-2) \cdot 180°.$$

  Note that $x - y < 90°$ and $0 < y < x < 180°$.

- Construct a polygon $B_1B_2 \dots B_k$ such that $B_1B_2 = B_2B_3 = \cdots = B_{k-1}B_k$, $\angle B_2 = \angle B_3 = \cdots = \angle B_{k-1} = x$, and $\angle B_1 = \angle B_k = y$.

- Take an isosceles triangle that has two base sides of length B_1B_k and two base angles equal to $x - y$. Attach outwards to each of the base sides a polygon congruent to $B_1B_2 \dots B_k$.

The resulting $(2k-1)$-gon has $2k-2$ congruent angles and $2k-2$ congruent sides, and is convex since $x < 180°$ and $2y + x < 180°$ by construction. But it is not equiangular, so it is not regular.

In particular, for $n = 5$, we can choose $x = 140°$ and $y = 40°$. We then start with an equilateral triangle and attach outwards to two of its sides two 40-100-40 isosceles triangles.

Alternate approach for first part. Draw a regular polygon $A_1A_2\ldots A_{2k-1}$ with side length 1. Let M be the midpoint of A_kA_{k+1}. Line A_1M is a line of symmetry for our polygon. Note that $\angle A_{2k-1}A_1A_2 = 180^\circ - \frac{360^\circ}{n} = \theta$ and $\angle MA_1A_2 = \theta/2$.

Construct point A_2' in the vicinity of A_2 such that $A_1A_2 = 1$ and $\angle MA_1A_2' = \theta/2 - (n-1)\varepsilon$, for some small $\varepsilon > 0$. Construct point $A_3', \ldots, A_k'$ in the vicinity of $A_3, \ldots, A_k$ such that $A_2'A_3' = \cdots = A_{k-1}'A_k' = 1$ and $\angle A_1A_2'A_3' = \angle A_1A_2'A_3' = \cdots = \angle A_{k-2}'A_{k-1}'A_k' = \theta + 2\varepsilon$. Symmetrically, construct points $A_1, A_{2k-1}', A_{2k-2}', \ldots, A_{k+1}'$ on the other side of the line A_1M. Because we can choose ε as small as we want, the resulting polygon $A_1A_2'\ldots A_{2k-1}'$ will be convex. It is not difficult to check that the constructed polygon has all angles, except at A_1, equal to $\theta + 2\varepsilon$, and all sides, except $A_k'A_{k+1}'$, have unit length. Polygon $A_1A_2'\ldots A_{2k-1}'$ can be viewed as a slightly distorted version of $A_1A_2\ldots A_{2k-1}$.

Second part. We now prove the statement is true for all even $n = 2k$. Suppose there is a polygon $A_1A_2\ldots A_n$ with

$$\angle A_2 = \angle A_3 = \cdots = \angle A_n = 180^\circ - \theta$$

and with sides of same length, except possibly the side A_pA_{p+1}.

Consider a coordinate plane with the origin O and define vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_n$ such that

$$\mathbf{v}_1 = \overrightarrow{A_1A_2}, \quad \mathbf{v}_2 = \overrightarrow{A_2A_3}, \quad \ldots, \quad \mathbf{v}_n = \overrightarrow{A_nA_1}.$$

Then obviously we have

$$\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_n = 0$$

since starting at A_1 and following the vectors ends up back at A_1. We are given that the angle between the vectors $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$ is equal to θ for any $1 \le i \le n-1$, and that all vectors have the same length except possibly $\mathbf{v}_p$.

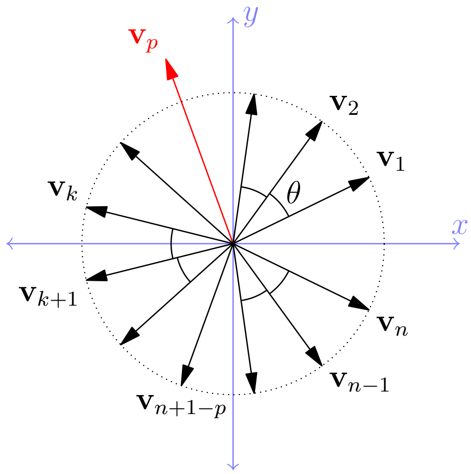


Figure 5: Vector setup with x and y axes with $\|\mathbf{v}_p\|$ unspecified.

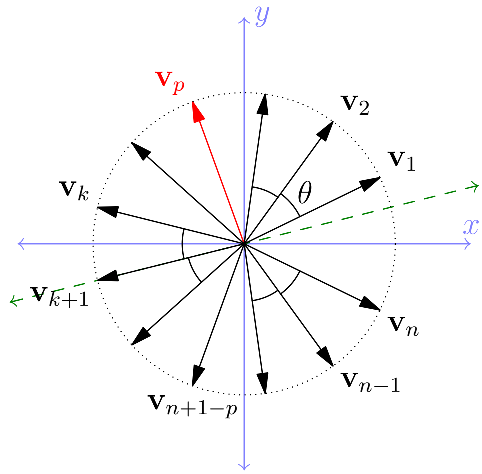


Figure 6: Projection onto $\mathbf{v}_{k+1}$ after having proven $\|\mathbf{v}_p\| = \|\mathbf{v}_{n+1-p}\|$.

We now impose axes on the figure: Place the x-axis such that it bisects the smaller angle between $\mathbf{v}_1$ and $\mathbf{v_n}$, with the endpoint of $\mathbf{v}_1$ in the first quadrant. This situation is illustrated in Figure 5.

Now the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$ can be split into pairs of vectors that are symmetric with respect to the x-axis, except possibly $\mathbf{v}_p$ and $\mathbf{v}_{n+1-p}$. The sum of the projections of the vectors on the y-axis must be equal to 0; in other words, the y-coordinates of $\mathbf{v}_p$ and $\mathbf{v}_{n+1-p}$ are equal. Since $\mathbf{v}_p$ and $\mathbf{v}_{n+1-p}$ have symmetric angles with respect to the y-axis and these projections have nonzero length, this forces $\mathbf{v}_p$ and $\mathbf{v}_{n+1-p}$ to have equal length. (So in fact, our original polygon must be equilateral).

Finally, consider the projection of all the vectors along the line ℓ through vector $\mathbf{v}_{k+1}$, as shown in Figure 6. As before the sum of the projections of ℓ on the x-axis equals 0. The projection of all vectors except $\mathbf{v}_1$ and $\mathbf{v}_{k+1}$ cancel out. But $\|\mathbf{v}_1\| = \|\mathbf{v}_{k+1}\|$, so the latter is possible if and only if $\mathbf{v}_{k+1} = -\mathbf{v}_1$, yielding the desired result.

Alternate approach for second part. One may also use complex numbers rather than vectors in the second part.

Suppose there is a polygon $A_1A_2 \dots A_n$ with

$$\angle A_1 = \angle A_2 = \cdots = \angle A_{n-1} = 180^\circ - \theta$$

and with sides of same length, say 1, except possibly for the side A_pA_{p+1}. (Note the indices differ by 1 from the previous solution.)

Let

$$z = e^{i\theta} = \cos\theta + i\sin\theta.$$

This time, we may impose complex coordinates such that

$$1 = \overrightarrow{A_nA_1}, \quad z = \overrightarrow{A_1A_2}, \quad z^2 = \overrightarrow{A_2A_3}, \quad \dots, \quad z^{n-1} = \overrightarrow{A_{n-1}A_n}$$

except that $\overrightarrow{A_pA_{p+1}}$ is equal to a real multiple of z^p rather than exactly equal to it; we denote this by $r \cdot z^p$ for $r \in \mathbb{R}$. This is illustrated in Figure 7.

Because of the convexity, we need $\theta < \frac{360^\circ}{n-1}$ (the complex numbers z^0, z^1, z^2 should have increasing argument since the original polygon was convex). We also have $z \neq 1$.

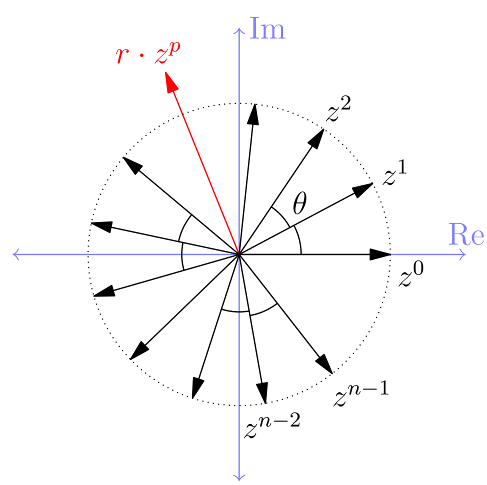


Figure 7: Construction for odd n.

As before, we have that the complex numbers here must sum to zero, so

$$\begin{aligned} 0 &= 1 + z + \cdots + z^{p-1} + rz^p + z^{p+1} + \cdots + z^{n-1} \\ &= [1 + z + \cdots + z^{n-1}] + (r-1)z^p \\ &= \frac{z^n - 1}{z - 1} + (r-1)z^p \\ \implies 1 - r &= \frac{z^n - 1}{z^p(z-1)}. \end{aligned}$$

Apparently, the right-hand side is a real number. So it should be equal to its complex conjugate. Since $|z| = 1$, we have $\bar{z} = 1/z$, so this occurs if

$$\frac{z^n - 1}{z^p(z-1)} = \frac{z^{-n} - 1}{z^{-p}(z^{-1} - 1)}.$$

Assume for contradiction now that $z^n \neq 1$ (otherwise we immediately have $r = 1$ and the entire problem is solved). Then the equation implies

$$z^n = z^{2p+1}.$$

Therefore, we have

$$(n - (2p+1)) \cdot \theta$$

is an integer multiple of 360°. But $n - (2p+1)$ has absolute value strictly less than n, and is nonzero since n is even. But $\theta < \frac{360^\circ}{n-1}$ and this is a contradiction.

Third approach for second part. One may instead study symmetry properties of the polygon in the second part.

We first prove the case $n = 4$. Suppose there is a polygon $A_1A_2A_3A_4$ with

$$A_1A_2 = A_2A_3 = A_3A_4 = 1$$

and with all angles of same measure, except possibly for the angle $\angle A_p$. Without loss of generality we can assume that $p \in \{1, 2\}$. We make the proof in the case $p = 1$; the case $p = 2$ is very similar. The polygon is therefore symmetric with respect to the perpendicular bisector of the side A_2A_3, because the symmetry exchanges A_2 and A_3, and since $\angle A_2 = \angle A_3$ it also exchanges the lines A_2A_1 and A_3A_4 and finally since $A_1A_2 = A_3A_4$ the symmetry exchanges A_1 and A_4. This symmetry means that $\angle A_1 = \angle A_4$ and hence the polygon is equiangular. Finally, we prove in a similar way that the polygon is also symmetric with respect to the angle bisector of $\angle A_2$ and therefore the polygon is equilateral, hence regular.

Now for $n \geq 6$, suppose there is a polygon $A_1A_2 \dots A_n$ with

$$A_1A_2 = A_2A_3 = \ldots = A_{n-1}A_n$$

and with all angles of same measure, except possibly for the angle $\angle A_p$. If p is either $1, \frac{n}{2}, \frac{n}{2} + 1$ or n, then we can as for the case $n = 4$ prove that the polygon is symmetric with respect to the perpendicular bisector of either A_nA_1 or $A_{\frac{n}{2}}A_{\frac{n}{2}+1}$ and therefore the polygon is equiangular. If p takes an other value, the diagonal A_pA_{n+1-p} separates the polygon into two polygons, which are symmetric with respect to the perpendicular bisector of A_nA_1 and $A_{\frac{n}{2}}A_{\frac{n}{2}+1}$ respectively. Combining the angle relations from these two polygons, we obtain $\angle A_p = \angle A_{n+1-p}$ and therefore the polygon is equiangular. Finally, as in the proof of $n = 4$ we prove that the polygon is symmetric with respect to the angle bisector of $\angle A_{\frac{n}{2}}$ and therefore the polygon is equilateral, hence regular.

Problem 7

Each of the n^2 cells of an $n \times n$ grid is colored either black or white. Let a_i denote the number of white cells in the i-th row, and let b_i denote the number of black cells in the i-th column. Determine the maximum value of $\sum_{i=1}^{n} a_i b_i$ over all coloring schemes of the grid.

Solution 1

The answer is $\frac{n^3-n}{3}$. We will refer to cells by their coordinates, i.e. (x, y) refers to the cell in row x and column y.

The maximum $\sum_{i=1}^{n} a_i b_i = \frac{n^3-n}{3}$ can be attained by coloring the cell (x, y) white if and only if $x \geq y$. With this coloring, we have $a_i = i$ and $b_i = i - 1$ for each i. Thus,

$$\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} i(i-1) = \frac{n^3-n}{3}.$$

It remains to show that no coloring can achieve more than $\frac{n^3-n}{3}$. To do this, consider the set S of all ordered triples (x, y, z) of integers with $1 \leq x, y, z \leq n$. Define the subsets

$$A := \{(x, y, z) \in S : (x, y) \text{ is black and } (y, z) \text{ is white}\}$$

$$B := \{(x, y, z) \in S : (y, z) \text{ is black and } (z, x) \text{ is white}\}$$

$$C := \{(x, y, z) \in S : (z, x) \text{ is black and } (x, y) \text{ is white}\}.$$

It is clear from their definition that these subsets are pairwise disjoint.

Moreover, for each $1 \leq i \leq n$, observe that the number of elements $(x, y, z) \in A$ for which $y = i$ is precisely $b_i \cdot a_i$ (there are b_i choices for x and a_i choices for z). Thus, $|A| = \sum_{i=1}^{n} a_i b_i$, and the same holds for B and C by symmetry.

Finally, we note that for each $1 \leq i \leq n$, the triple (i, i, i) is not in any of A, B, or C, since the cell (i, i) cannot be both white and black. Thus, we have

$$|A| + |B| + |C| + n \leq |S|$$

and hence

$$3 \sum_{i=1}^{n} a_i b_i \leq n^3 - n,$$

which rewrites to

$$\sum_{i=1}^{n} a_i b_i \leq \frac{n^3-n}{3},$$

as desired.

Solution 2

As in the previous solution, we refer to the cell in row x and column y by (x, y). We construct a configuration achieving the bound $\frac{n^3-n}{3}$ as in Solution 1. To prove that this is best possible, define an $n \times n$ matrix (x_{ij}) by setting $x_{ij} = 1$ if (i, j) is white and $x_{ij} = 0$ if (i, j) is black; then

$$\sum_{i=1}^{n} a_i b_i = \sum_{i,j,k=1}^{n} x_{ij}(1 - x_{ki});$$

By relabeling indices in the sum, we obtain

$$3\sum_{i=1}^{n} a_i b_i = \sum_{i,j,k=1}^{n} (x_{ij}(1 - x_{ki}) + x_{jk}(1 - x_{ij}) + x_{ki}(1 - x_{jk})).$$

When $i = j = k$, the summand $x_{ij}(1-x_{ki})+x_{jk}(1-x_{ij})+x_{ki}(1-x_{jk})$ equals 0 because each of the three products includes a zero factor. Otherwise, one can see in several ways that the summand can never exceed 1; for example, by cyclic symmetry we only have to consider the possibilities

$$(x_{ij}, x_{jk}, x_{ki}) = (0, 0, 0), (0, 0, 1), (0, 1, 1), (1, 1, 1),$$

and in these cases the summands equals respectively 0, 1, 1, 0.

Solution 3

As in the previous solution, we refer to the cell in row x and column y by (x, y). Starting from the given configuration, we perform a series of operations that do not decrease $\sum_{i=1}^{n} a_i b_i$, at the end of which we have $\sum_{i=1}^{n} a_i b_i = \frac{n^3-n}{3}$; this will establish the upper and lower bounds at the same time.

1. Apply a common permutation to the rows and to the columns to ensure that $a_1 \leq \cdots \leq a_n$. This does not change $\sum_{i=1}^{n} a_i b_i$.

2. Repeat the following operation as many times as possible: find two indices i, j such that (i, j) is black and $(i, j + 1)$ is white, and exchange the colors of these two cells. This decreases b_j by 1 and increases b_{j+1} by 1 without changing any of the other counts, so $\sum_{i=1}^{n} a_i b_i$ increases by the amount $a_{j+1} - a_j \geq 0$ (and the condition $a_1 \leq \cdots \leq a_n$ remains true).

   To see that this process eventually terminates, count the number of pairs of cells in the same row (not necessarily adjacent) consisting of a black cell to the left of a white cell; this number is evidently nonnegative and decreases by 1 at each step, so the process cannot continue indefinitely.

   We now have a configuration in which in each row, all white cells are to the left of all black cells, and (by that condition plus the condition $a_1 \leq \cdots \leq a_n$) in each column, all black cells are above all white cells. For short, we characterize such a configuration as being *echelonized*.

3. Repeat the following operation as many times as possible: choose the largest index i for which $a_i \leq i-2$, set $j = a_i + 1$, and change the color of the cell (i, j) from black to white. (Note that the maximality of i ensures that if $i < n$, then the cell $(i+1, j)$ is white, so the resulting configuration is still echelonized; see the left figure below.) This has the effect of increasing a_i by 1 and decreasing b_j by 1 without changing any of the other counts, so $\sum_{i=1}^n a_i b_i$ increases by the amount

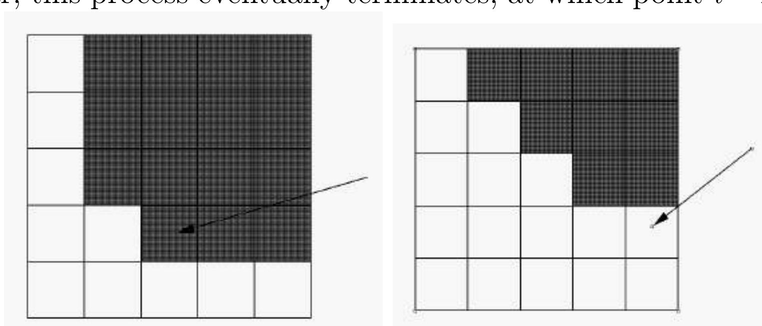
$$b_i - a_j \geq b_j - a_i \geq i - a_i > 0.$$

In particular, this process eventually terminates, at which point $a_i \geq i-1$ for all i.

4. Repeat the following operation as many times as possible: choose the smallest index i for which $a_i \geq i+1$, set $j = a_i$, and change the color of the cell (i, j) from white to black. (Note that the minimality of i ensures that if $i > 0$, then the cell $(i-1, j)$ is black, so the resulting configuration is still echelonized; see the right figure below.) This has the effect of decreasing a_i by 1 and increasing b_j by 1 without changing any of the other counts, so $\sum_{i=1}^n a_i b_i$ increases by the amount

$$a_j - b_i \geq a_i - b_j \geq a_i - (i-1) > 0.$$

In particular, this process eventually terminates, at which point $i - 1 \leq a_i \leq i$ for all i.



5. For $i = 1, \dots, n$, if (i, j) is black, change it to white. This does not change $\sum_{i=1}^n a_i b_i$.

We thus arrive at the configuration in which the cell (i, j) is colored black if $i < j$ and white if $i \geq j$. In this configuration, we compute that

$$\sum_{i=1}^n a_i b_i = \sum_{i=1}^n i(i-1) = \frac{n^3 - n}{3},$$

completing the proof.

Remark. Given that $a_1 \leq \cdots \leq a_n$, one can then swap columns to ensure that $b_1 \leq \cdots \leq b_n$, and this does not decrease $\sum_{i=1}^n a_i b_i$ thanks to the rearrangement inequality. However, this is not sufficient to guarantee that the resulting configuration is echelonized.

Solution 4

As in the previous solutions, we refer to the cell in row x and column y by (x, y). We prove the bound of $(n^3 - n)/3$ independently.

Consider a coloring of the grid for which the quantity $S = \sum_{i=1}^n a_i b_i$ is maximized, and among those colorings, consider one for which the quantity $T = \sum_{i=1}^n a_i^2$ is maximized. Without loss of generality, we may swap the rows and columns of the grid so that the a_i are non-decreasing; that is, $a_1 \leq a_2 \leq \cdots \leq a_n$. We now prove some facts about this maximal grid.

Fact 1: If $a_i = a_j$, then (i, y) and (j, y) are the same color for all y.

Proof: Assume otherwise. Since rows i and j have the same number of white cells, there must be some y_1 and y_2 for which (i, y_1) and (j, y_2) are white, and (i, y_2) and (j, y_1) are black.

Now, consider swapping the color on the cells (i, y_1) and (j, y_1). Then, a_i increases by 1, a_j decreases by 1, and all other values of a and b stay the same. Thus, S increases by $b_i - b_j$, and hence $b_i \leq b_j$. By similar reasoning, $b_i \geq b_j$, so we have $b_i = b_j$. Hence, we may swap the colors on the cells (i, y_1) and (j, y_1), which does not change S but increases T by 2, contradicting the maximality assumption that we had previously. ■

Fact 2: If $a_i < a_j$, then we cannot have that (i, y) is black but (j, y) is white.

Proof: As in the previous fact, assume that (i, y) is black but (j, y) is white. Then, swapping the two increases S by $a_j - a_i$, which contradicts the maximality assumption. ■

Due to Facts 1 and 2 and our assumption that the a_i are non-decreasing, we know that for each y, there is some value b_y such that the cells (i, y) are white when $i \leq n - b_y$ and they are black when $(i, y) \geq n - b_y + 1$. Furthermore, each choice of $b_1, b_2, \ldots, b_n$ now defines a coloring of the grid. Clearly this matches the original definition of b_y we have had. Furthermore, we have that a_i is equal to the number of values y for which $b_y \leq n - i$. Thus, we have

$$\sum_{i=1}^n a_i b_i = \sum_{i=1}^n b_i |\{j | b_j \leq n - i\}|.$$

Since the value of $|\{j | b_j \leq n - i\}|$ are in non-increasing order, we may assume without loss of generality that the b_i are also in non-increasing order by the rearrangement inequality. We now inductively prove that $b_{n-i} = i$; that is, we prove:

Claim: Assume that for $0 \leq i < m$, we have that $b_{n-i} = i$ in our maximal grid. Then, we must have that $b_{n-m} = m$ too.

Proof: We first note that

$$\sum_{i=1}^n b_i |\{j | b_j \leq n-i\}| = \sum_{i=1}^{n-m} b_i (m + |\{j | 1 \leq j \leq m, b_j \leq n-i\}|) + m|\{j | 1 \leq j \leq m, b_j = m-1\}| + \sum_{i=0}^{m-1} i^2.$$

Now, consider decreasing the value of b_{n-m} from k to $k-1$. Then, the sum above changes by

$$b_{n-k+1} - m - |\{j | 1 \leq j \leq m, b_j \leq m\}|$$

whenever $k \neq m+1$, and the sum above changes by

$$-|\{j | 1 \leq j \leq m-1, b_j \leq m\}|$$

when $k = m+1$, because the only terms affected in the sum are when $i = n - k + 1$ and when $i = n - m$.

Assume now that in our maximal grid, $b_{n-m} \geq m+1$. Then, if $b_{n-m} > m+1$, we have that $\{j | 1 \leq j \leq m, b_j \leq m\} = \varnothing$ because the b_i are non-increasing, and so we can increase the sum by $b_{n-k+1} - m > 0$ by decreasing b_{n-m} by 1. If $b_{n-m} = m+1$, then the sum remains unchanged if we decrease b_{n-m} by 1, but this would increase some a_k by 1, so it would keep $S = \sum_{i=1}^{n} a_i b_i$ the same but increase $T = \sum_{i=1} a_i^2$. Thus, it cannot be that $b_{n-m} \geq m+1$ in our maximal grid.

Now, assume that in our maximal grid, $b_{n-m} = m-1$. Then, when we increase b_{n-m} to m, S changes by $m + |\{j | 1 \leq j \leq m, b_j \leq m\} - b_{n-m+1}$, which is positive by assumption that $b_{n-m+1} = m-1$. Hence, we have created a grid (where the b_i are possibly no longer sorted) that increases our value of S, which is again a contradiction.

Thus, we have established that $b_{n-m} = m$. ■

From the claim, we have now established that $b_{n-i} = i$ for each $i = 0, 1, \ldots, n-1$, in which case we may compute $S = (n^3 - n)/3$ for our maximal configuration, as desired.

Solution 5

We give another proof of the bound $(n^3 - n)/3$. (To prove that this bound is achieved, we use the same construction as in the previous solutions.)

Note that $a_i b_i \leq \lfloor \frac{(a_i+b_i)^2}{4} \rfloor$ by AM-GM. For brevity, write $s_i = a_i + b_i$. The sum $s_1 + \cdots + s_i$ counts each square in the union of the first i rows and the first i columns at most once; because this union contains $2ni - i^2$ squares, it follows that $s_1 + \cdots + s_i \leq 2ni - i^2$. We will use these estimates to bound $\sum_{i=1}^{n} \lfloor \frac{s_i^2}{4} \rfloor$. In particular, we make the following claim:

Claim. If $s_1 \geq s_2 \geq \cdots \geq s_n \geq 0$ are integers such that $s_1 + \cdots + s_i \leq 2ni - i^2$ for $i = 1, 2, \cdots, n$, then $S = \lfloor \frac{s_1^2}{4} \rfloor + \cdots + \lfloor \frac{s_n^2}{4} \rfloor$ is maximized (at least) when $s_i = 2(n-i)+1$ for all i.

Proof of claim. We describe an operation that can be applied in all cases except when $s_i = 2(n-i)+1$ for all i, which never decreases S. Since S is trivially bounded above (because each s_i is), after finitely many steps we must arrive at the case where $s_i = 2(n-i)+1$ while having not decreased S, proving the claim.

Note that $s_1 + \cdots + s_k \leq 2nk - k^2$ becomes an equality when $s_i = 2(n-i)+1$ for $i = 1$, 2, $\ldots$, k. Therefore, if i is the smallest index for which $s_i \neq 2(n-i)+1$, we necessarily have $s_i < 2(n-i)+1$ and $s_1 + \cdots + s_i < 2ni - i^2$. If there exists an index $j > i$ such that $s_j > s_{j+1}$, we choose the smallest such j and replace s_i by $s_i + 1$ and s_j by $s_j - 1$; otherwise, if $s_n > 0$, we replace s_i by $s_i + 1$ and s_n by $s_n - 1$ (we define $j = n$ in this case); otherwise, we replace s_i by $s_i + 1$ and do nothing else. It is clear that the resulting sequence $s_1, \ldots, s_n$ still satisfies $s_1 \geq s_2 \geq \cdots \geq s_n \geq 0$.

It remains to show that: (1) S never decreases when we apply the operation, and (2) the inequalities $s_1 + \cdots + s_i \leq 2ni - i^2$ continue to hold.

For (1), note that from $\lfloor x^2/4 \rfloor \geq (x^2-1)/4$ it follows that if $j < n$, then

$$\left\lfloor \frac{(s_i+1)^2}{4} \right\rfloor + \left\lfloor \frac{(s_j-1)^2}{4} \right\rfloor \geq \frac{s_i^2}{4} + \frac{s_j^2}{4} + \frac{s_i - s_j}{2} \geq \left\lfloor \frac{s_i^2}{4} \right\rfloor + \left\lfloor \frac{s_j^2}{4} \right\rfloor;$$

so S can never increase, as desired.

For (2), note that $s_1 + \cdots + s_k \leq 2nk - k^2$ remains true for $k < i$ and for $k \geq j$ if j is defined (because for these values $s_1 + \cdots + s_k$ remains unchanged). We now show that the inequality $s_1 + \cdots + s_k \leq 2nk - k^2$ also remains true for $i \leq k < j$ (or for all $i \leq k$ if j is undefined). To this end, we show that $s_1 + \cdots + s_k < 2nk - k^2$ holds before increasing s_i by 1; then surely $s_1 + \cdots + s_k \le 2nk - k^2$ must holds after increasing s_i by 1. Suppose for the sake of contradiction that $s_1 + \cdots + s_k = 2nk - k^2$, i.e. that we already have equality in $s_1 + \cdots + s_k \le 2nk - k^2$ before increasing s_i by 1.

- If j is defined, then $s_{i+1} = s_{i+2} = \cdots = s_k = \cdots = s_j = s$. We know that $2nj - j^2 \ge s_1 + \cdots + s_j = (s_1 + \cdots + s_k) + (k-j)s = 2nk - k^2 + (k-j)s$, from which it follows that $s \le 2n - j - k$. However, we then obtain $s_1 + \cdots + s_k = (s_1 + \cdots + s_i) + (k-i)s < 2ni - i^2 + (k-i)(2n-j-k) \le 2ni - i^2 + (k-i)(2n-i-k) = 2nk - k^2$, contradicting the assumption that $s_1 + \cdots + s_k = 2nk - k^2$.

- If j is undefined, we have $s_{i+1} = \cdots = s_n = 0$; so then simply $s_1 + \cdots + s_k = s_1 + \cdots + s_i < 2ni - i^2 \le 2nk - k^2$ shows that we cannot have equality in $s_1 + \cdots + s_k \le 2nk - k^2$ prior to increasing s_i by 1.

This concludes the proof of (2), and thereby the proof of the claim.

After reordering the a_i and b_i such that (s_i) is non-increasing, we can apply the claim to find that

$$\sum_{i=1}^{n} a_i b_i \le \sum_{i=1}^{n} \left\lfloor \frac{(2(n-i)+1)^2}{4} \right\rfloor = \sum_{i=1}^{n} (n-i+1)(n-i) = \sum_{i=1}^{n} i(i-1) = \frac{n^3 - n}{3},$$

as desired.

Solution 6

Let (i,j) denote the cell in the ith row and jth column. The approach of this solution is to show that in a coloring where $T = \sum_{i=1}^{n} a_i b_i$ is maximized, (i,j) and (j,i) are colored differently for all $i \ne j$. We will refer to (i,j) and (j,i) as *reflected pairs* for $i \ne j$. For $1 \le k \le n$, let S_k denote the set of cells (i,j) with $i = k$ or $j = k$ but $i \ne j$.

Note that in an optimal coloring, if S_k contains more black cells than white cells then (k,k) must be colored white, and vice versa. Indeed, suppose that outside of (k,k), the kth row contains u white cells and the kth column contains v white cells. The contribution of $a_k b_k$ to T is $u(n-v)$ if (k,k) is colored black and $(u+1)(n-1-v)$ if (k,k) is colored white. Under the assumption that there are more black cells than white cells in S_k, we have that $u + v < n - 1$. Thus, $u(n-v) - (u+1)(n-1-v) = u + v - n + 1 < 0$, so (k,k) must be colored white. The analogous statement holds when we switch the colors. Similarly, if S_k contains exactly $n-1$ white cells and $n-1$ black cells, then the color of (k,k) may be arbitrary without impacting the value of T. Henceforth, we will assume that if (k,k) is colored with whichever color appears less often in S_k. If S_k contains an equal amount of black and white, then we may freely decide the color of (k,k) to aid us in our solution.

Suppose that there exists a reflected pair of cells of the same color. We will demonstrate a procedure to recolor one of the cells in a reflected pair that does not decrease T. By applying this repeatedly, we will eventually arrive at a coloring in which all reflected pairs are colored differently. Consider a S_k which has a maximal difference between the number of white and black cells. Without loss of generality, suppose that there are at least as many white cells as black cells. If S_k has strictly more white cells than black cells, then at least one of the $n-1$ reflected pairs in S_k must have both cells colored white. If S_k has an equal number of black and white cells, then this must be true of all S_ℓ. In that case, we will consider any monochromatic reflected pair of cells. Suppose that this reflected pair of cells is (k,ℓ) and (ℓ,k), both colored white. By recoloring these cells, we only change the values of $a_k b_k$ and $a_\ell b_\ell$ in T. Let r_k, r_ℓ, c_k, c_ℓ be the number of white cells in the kth row, ℓth row, kth column, and ℓth column, respectively. We have that (k,k) is colored black by remarks at the end of the previous paragraph. If (k,ℓ) is colored black, then $a_k b_k$ decreases by $n-c_k$ and $a_\ell b_\ell$ increases by r_ℓ, for a total change of $r_\ell + c_k - n$. Similarly, if (ℓ,k) is colored black, then T changes by $r_k + c_\ell - n$. If one of these is nonnegative, we may switch the color of the corresponding cell, as desired. Otherwise, we must have that $c_k < n - r_\ell$ and $r_k < n - c_\ell$. Note that $c_k + r_k$ is the number of white cells in S_k since (k,k) is black, and that there are at least $n - r_\ell - 1 + n - c_\ell - 1$ black cells in S_ℓ, since $n - r_\ell$ and $n - c_\ell$ respectively count the number of black cells in the ℓth row and ℓth column. Hence, the number of black cells in S_ℓ is at least the number of white cells in S_k, which means that we may assume that (ℓ,ℓ) is colored white. But then, the number of black cells in S_ℓ is strictly more than the number of white cells in S_k, which contradicts the maximality of our choice of S_k. Thus, we may always recolor one of (k,ℓ) and (ℓ,k), and repeating this process results in a coloring where reflected pairs are colored differently, as desired.

To finish, we first note that the colors on the diagonal of cells (k,k) do not impact the value of T. Consider a tournament on vertices $v_1,\ldots,v_n$, where $v_i \to v_j$ if and only if (i,j) is colored white. Let d_i denote the outdegree of v_i. We now have that $T = \sum_{i=1}^n d_i(d_i + 1) = 2\sum_{i=1}^n d_i + 2\sum_{i=1}^n \binom{d_i}{2} = n(n-1) + 2\sum_{i=1}^n \binom{d_i}{2}$. The quantity $\sum_{i=1}^n \binom{d_i}{2}$ counts the number of ordered triples (v_i, v_j, v_k) of distinct vertices with edges $v_i \to v_j$ and $v_i \to v_k$. Each unordered triple of vertices contributes at most 1 such ordered triple. It follows that $T \le n(n-1) + 2\binom{n}{3} = \frac{n^3-n}{3}$. Equality holds when the tournament is transitive, for example achieved when (i,j) is colored white whenever $i \ge j$ and black otherwise.

Problem 8

Let $a_1, a_2, \ldots$ be an infinite sequence of positive real numbers such that for each positive integer n we have

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_{n+1}^2}{n+1}}.$$

Prove that the sequence $a_1, a_2, \ldots$ is constant.

Solution

Define

$$S_n = \frac{a_1 + a_2 + \ldots + a_n}{n} \quad \text{and} \quad Q_n = \sqrt{\frac{a_1^2 + a_2^2 + \ldots + a_n^2}{n}}.$$

Fix an integer $k \geq 1$. For all $n \geq k$ the problem condition gives us that

$$\begin{aligned}
Q_n^2 - Q_{n+1}^2 \geq Q_n^2 - S_n^2 &= \frac{\sum_{1 \leq i < j \leq n} (a_i - a_j)^2}{n^2} \\
&\geq \frac{\sum_{k \leq j \leq n} (a_k - a_j)^2 + (a_j - a_1)^2}{n^2} \\
&\geq \frac{\sum_{k \leq j \leq n} \frac{1}{2}((a_k - a_j) + (a_j - a_1))^2}{n^2} \\
&= \frac{(n-k+1)}{2n^2}(a_k - a_1)^2.
\end{aligned}$$

Summing the previous inequality over all $n \geq k$ gives that

$$Q_k^2 \geq (a_k - a_1)^2 \sum_{n \geq k} \frac{(n-k+1)}{2n^2}.$$

Because the sum $\sum_{n \geq k} \frac{(n-k+1)}{2n^2}$ diverges, we get that $a_k = a_1$ as desired.